GRADIENT ESTIMATES IN L^p FOR SOLUTIONS OF NONLINEAR PARABOLIC EQUATIONS WITH m-LAPLACIAN TYPE

NAIMAH ARIS[1] and JEFFRY KUSUMA[2]

ABSTRACT. This paper discusses the gradient estimates for solution of nonlinear parabolic equations with m-Laplacian type:

$$u_t - div(\sigma(|\nabla u|^2)\nabla u) + g(u) = f(x), \qquad t > 0,\ \ x \in \Omega,$$

where $\sigma(v^2)$ is a function like $\sigma(v^2) = |v|^m$, function $\sigma(\cdot)$ is differentiable on R^+, $g(u)$ is a globally Lipschitz function on R with $g(0) = 0$, f belongs to $L^P(\Omega)$ and Ω is a bounded domain in R^n.

Key words: Nonlinear parabolic equations, m-Laplacian type, gradient estimates.

1. Introduction

We consider a typical strongly nonlinear parabolic equation so-called m-Laplacian type. The equation we consider is the following:

$$u_t - div(\sigma(|\nabla u|^2)\nabla u) + g(u) = f(x) \qquad t > 0, x \in \Omega, \tag{1.1}$$

with

$$u(x,0) = u_0(x), \quad x \in \Omega; \quad u(x,t) = 0, \quad x \in \partial\Omega, t > 0 \tag{1.2}$$

where $\sigma(v^2)$ is a function like $\sigma(v^2) = |v|^m$ and $g(u)$ is a globally Lipschiz function . This is one of the most typical nonlinear parabolic equations, investigated quite often from various points of view (Haraux[5], Cholewa and Dlotko [3], Nakao, Chen[8], etc).

When $\sigma(\mathrm{v}) = |v|^{\frac{p-2}{2}}$, $p \geq 2$, Alikakos and Rostamin [1] derived an estimate $\| \nabla u(t) \|_\infty$ for the solutions of equations with Neumann boundary condition, which includes a smoothing effect and decay properties. In Nakao, Aris [7] also [1], a strong coerciveness condition on $-div(\sigma(|\nabla u|^2)\nabla u)$ is used with a mean curvature type $\sigma(v^2) = \frac{1}{\sqrt{1+v^2}}$ is excluded.

Engler, Kawohl and Luckhaus[2] have treated the problem (1)-(2) with $g(u) = f(x) = 0$ and derive estimates for $\| \nabla u(t) \|_q$, for any $q \geq 2$. Nakao, Ohara [9], derive gradient estimates as well as global existence of (1)-(2) with an convection term $\mathbf{b}(\mathrm{u}) \cdot \nabla \mathrm{u}$.

The main purpose of this paper is to derive the gradient estimate in $L^\mathrm{p}(\Omega)$ spaces of the equations (1)-(2) . For the proof, we use Moser's technique as in Nakao[6], Nakao and Chen [9], Nakao and Ohara[11]. Theoritical consideration of the following section are mostly based on references [4] and [10].

[1] Jurusan Matematika FMIPA Universitas Hasanuddin

2. Preliminaries and results

The function spaces we use are all standard and the definition of them are omitted. But we note that $\|\cdot\|_p$, $0 \le p < \infty$, denotes L^P norm on Ω.

Here, we assume $\partial\Omega$ is C^2-class and make the following assumptions:

Assumption A

$\sigma(\cdot)$ is differentiable on $R^+ = [0, \infty)$ and satisfies the conditions:

$$k_0|v|^m \le \sigma(v^2) \; and \; k_0|v|^m \le \sigma'(v^2)v^2$$

and

$$k_1\sigma(v^2)v^2 \le \int_0^{v^2} \sigma(\eta)d\eta$$

for $v \in R$ where $m \ge 0$ and k_0, k_1 are some positive constants.

Assumption B

$g(u)$ is a globally Lipschitz function on R with $g(0) = 0$.

Assumption C

f belongs to $L^P(\Omega)$. (Set $M = \| f \|_\infty$)

.

As a definiton of solution for (1)-(2) we employ definition

Definition 2.1.

A measurable function $u(x,t)$ on $\Omega \times R$ is called the solution of the problem (1.1)-(1.2) iff

$$u(t) \in L^2_{loc}([0,\infty); W_0^{1,p}(\Omega))$$

and,

$$\int_\infty^0 \int_\omega (-u_t(u,t)\Phi(u,t) + \sigma(|\nabla u|^2)\nabla u\Phi(u,t) + \lambda u\Phi(u,t) + g(x,u)\Phi(u,t) - f(x)\Phi(u,t))dxdt = 0$$

For the proof the following well-known inequalities play an essential role.

Lemma 2.1. (Gagliardo-Nirenberg)

Let $\beta \ge 0, N > p \ge 1, \beta + 1 \le q$ and $1 \le r \le q \le (\beta+1)Np/(N-p)$. Then for u such that $|u|^\beta u \in W^{1,p}(\Omega)$, we have

$$\| u \|_q \le C^{1/(\beta+1)} \| u \|_r^{1-\theta} \| |u|^\beta u \|_{1,p}^{\theta/(\beta+1)}$$

with $\theta = (\beta+1)(r^{-1} - q^{-1})/(N^{-1} - p^{-1} + (\beta+1)r^{-1})$, and C is a constant independent of q, r, β and θ if $N \ne p$, and the constant depending on $q/(\beta+1)$ if $N = p$.

Lemma 2.2.

Let $y(t)$ *be a nonnegative differentiable function on* $(0,T)$ *satisfying*

$$y'(t)+At^{\lambda\theta-1}y^{1+\theta}(t)\le Bt^{-k}y(t)+Ct^{-\delta}$$

with $A,\theta>0,\lambda\theta\ge 1,B,C\ge 0,k\le 1$. *Then, we have*

$$y(t)\le A^{-1/\theta}(2\lambda+2BT^{1-k})^{1/\theta}t^{-\lambda}+2C(\lambda+BT^{1-k})^{-1}t^{1-\delta},0<t\le T.$$

Lemma 2.3.

Let $y(t)$ *be a differentiable function on* $[1,\infty)$ *such that*

$$y'(t)+Ay^{1+\theta}(t)\le By(t)+C,1\le t$$

for some $A,B,C>0$. *Then*

$$y(t)\le max\{1,(A^{-1}(B+C)^{1/\beta_n}\},t\ge 1.$$

Lemma 2.4.

Let $y(t)$ *be a differentiable function on* $[1\infty)$ *satisfying*

$$y'(t)+At^{\mu}y^{1+\theta}(t)\le Bt^{-k},\quad 1\le t$$

for some $A,B,\mu>0,k\ge 0$. *Then*

$$y(t)\le Ct^{-\gamma},\quad t\ge 1$$

with some $C>0$ *independent of* y *and* $\gamma=min\{(1+\mu)//\theta,(\mu+k)/(1+\theta)\}$.

Lemma 2.5

Let $p_1\ge 1$ *and define* p_n *inductively by* $p_n=Rp_{n-1}-m$ *with* $R>1,m>0$. *Further, we set*

$$\theta_n=NR(1-p_{n-1}p_n^{-1})(N(R-1)+r)^{-1}$$

and

$$\beta_n=(p_n+m)\theta_n^{-1}-p_n,$$

for $n=2,3,\cdots$, *where* $r>0$. *Finally, for given* $\lambda_1\ge 0$, *we define* $\{\lambda_n\}$ *by*

$$\lambda_n=(1+\lambda_{n-1}(\beta_n-m))\beta_n^{-1},$$

for $n=2,3,\cdots$. *Then, we have*

$$\lim_{n\to\infty}\lambda_n=\frac{p_1\lambda_1 r+N}{p_1 r+mN}.$$

2.1. Estimates for $\| \boldsymbol{u}(t) \|_2$ and $\| \boldsymbol{u}(t) \|_\infty$

We give a priori estimates for the solutions. The solutions are in fact given as limits of smooth solutions of appropriate approximate equations and we may assume for our argument that the solutions under consideration are sufficiently smooth.

Proposition 2.1. *Let* $u(t)$ *be a solution of the problem* (1.1)-(1.2). *Then we have*

$$\| u(t) \|_2\le C(M_0,\| u_0 \|_2),\qquad 0\le t<\infty, \tag{2.1}$$

and

$$\| u(t) \|_\infty \le C(M_0, \| u_0 \|_2) t^{-\lambda}, \quad 0 < t \le 1, \tag{2.2}$$

Proof. The estimate (2.1) can be proved by multiplying the equation (1.1) by $u(t)$ and integrating on Ω. Indeed, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega \sigma(|\nabla u|^2)|\nabla u|^2 + \int_\Omega g(u) u dx = \int_\Omega f(x) u dx$$

Under our assumptions, we have

$$\frac{1}{2}\frac{d}{dt} \| u(t) \|_2^2 + \frac{k_0}{2} \| \nabla u \|_{m+2}^{m+2} \le C(M_0 \| u(t) \|_2 + \| u(t) \|_2^2). \tag{2.3}$$

Since $\| \nabla u \|_{m+2} \le C \| u \|_2$ (2.3) implies immediately (2.1).

Next, multiplying the equation (1.1) by $u_t(t)$ and integrating we have

$$\frac{1}{2}\int_\Omega u_t^2 dx + \int_\Omega \sigma(|\nabla u|^2)\nabla u \nabla u_t dx + \int_\Omega g(u) u_t dx = \int_\Omega f(x) u_t dx.$$

To prove (2.2) we multiply the equation (1.1) by $|u|^{p-2}u, p \ge 0$, and integrate by parts on Ω to get

$$\frac{1}{p}\frac{d}{dt} \| u(t) \|_p^p + (p-1)\int_\Omega \{\sigma |u|^{p-2}|\nabla u|^2 + g(u)|u|^p u) dx = \int_\Omega f|u|^p u dx.$$

Under our assumptions we have

$$\frac{1}{p}\frac{d}{dt} \| u(t) \|_p^p + \varepsilon_0 (p-1)\left(\frac{m+2}{p+m+2}\right)^{m+2} \| \nabla(|u|^{(p+m)/(m+2)} u) \|_{m+2}^{m+2}$$
$$\le C \| u(t) \|_p^p + C M_0 \| u(t) \|_p^{p-1} \tag{2.4}$$

We take $p_1 = m$ and $p_n = (m+2)p_{n-1} - m, n = 2,3. \cdots$.
Then, by Gagliardo-Nirenberg inequality,

$$\| u \|_{p_n} \le C^{(m+2)/(p_n+m)} \| u \|_{p_{n-1}}^{1-\theta_n} \| u^{(p_n+m)/(m+2)} \|_{1,m+2}^{(m+2)\theta_n/(p_n+m)} \tag{2.5}$$

with

$$\theta_n = (m+2)N(1 - p_{n-1}p_n^{-1})/(m+2+N(m+1)).$$

Therefore we see from (2.4) that

$$\frac{d}{dt} \| u(t) \|_{p_n} + C^{-\frac{m+2}{\theta_n}} p_n^{-1-m} \| u \|_{p_{n-1}}^{1+\beta_n} \| u \|_{p_n}^{1+\beta_n} \le C \| u(t) \|_{p_n} + C M_0. \tag{2.6}$$

Applying Lemma 2.2 to (2.6) and by induction we have

$$\| u(t) \|_{p_n} \le \eta_n t^{-\lambda_n}, \; 0 < t \le 1, \tag{2.7}$$

with

$$\lambda_n = (\beta_n - m)\lambda_{n-1} + 1)/\beta_n, \; \textit{and} \; \eta_n = \eta_{n-1}(C^{(m+2)/\theta_n}\lambda_n p_n^{m+1})^{1/\beta_n}.$$

We easily see that $\{\eta_n\}$ is bounded and further , by Lemma 2.1,

$$\lim_{n\to\infty} \lambda_n = \lambda N/(mN + 2(m+2)). \blacksquare$$

Proposition 2.2. *Let* $u(t)$ *be the solution of the problem* (1.1)-(1.2), *we have*

$$\| u(t) \|_{\infty} \leq C(M_0, \| u_{0_2}), \quad t \geq 1, \tag{2.8}$$

Proof. We return to the inequality (2.6). Then , applying Lemma 2.3 inductively we have

$$\chi_n \leq \max\{\| u(1) \|_{\infty}, 1, [(\frac{p_n + m}{m+2})^{m+2} C(M_0)(\varepsilon_1(p_n - 1))\chi_{n-1}\beta_n]^{\frac{1}{\beta_n}}\}, \tag{2.9}$$

where we set $\chi_n \equiv \sup_{t\geq 1} \| u(t) \|_{p_n}$. Applying Lemma 4.3 we can show that $\{\chi_n\}$ is bounded, that is

$$\chi_n \leq C(\| u(1) \|_{\infty}, M_0) < \infty.$$

Thus, combining this with the estimate (2.2) we obtain

$$\sup_{t\geq 1} \| u(t) \|_{\infty} \leq \limsup_{\chi_n} \leq C(M_0) < \infty. \blacksquare$$

2.2. Gradient Estimates in L^{P}

Estimate for $\| \nabla \boldsymbol{u}(\boldsymbol{t}) \|_{\boldsymbol{m+2}}$

Proposition 2.3. *Let* $u(t)$ *be a solution of problem (1.1)-(1.2). Then we have*

$$\| \nabla u(t) \|_{m+2} \leq C(M_0, \| u_0 \|_2) t^{-\lambda}, \quad 0 < t \leq 1, \tag{2.10}$$

for a certain $\lambda > 0$, and

$$\| \nabla u(t) \|_{m+2} \leq C(M_0, \| u_0 \|_2) < \infty, \quad t \geq 1. \tag{2.11}$$

Proof. Multiplying (1.1) by u_t, we see

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Gamma(|\nabla u|^2) + \| u_t(t) \|_2^2 &= \int_\Omega (g(u) + f)u_t\, dx \\ &\leq \frac{1}{2} \| u_t \|^2 + 2(\| f(t) \|^2 + \| u(t) \|^2) \end{aligned} \tag{2.12}$$

where

$$\Gamma(|\nabla u|^2) = \int_\Omega \int_0^{|\nabla u|^2} \sigma(s) ds dx.$$

Next, multiplying (1.1) by u we have

$$\begin{aligned} \Gamma_\varepsilon(|\nabla u|^2) &\leq C \int_\Omega \sigma(|\nabla u|^2)|\nabla u|^2\, dx \\ &= C \int_\Omega (-u_t - g(u) + f)u\, dx C \leq C \| u_t \|_2 (\| u \|_2 + \| f \|). \end{aligned} \tag{2.13}$$

From (2.12) and (2.13) we have

$$\Gamma \leq C(-\frac{d}{dt}\Gamma(t) + M_0^2 + \| u(t) \|^2)^{1/2}(\| u(t) \| + M_0)$$

and hence,

$$\frac{d}{dt}\Gamma(t) + \Gamma(t)^2 \leq C(\| u(t) \|^2 + M_0^2)$$

Combining this with the estimates for $\| u(t) \|$ we obtain (2.10) and (2.11). ■

Estimate for $\| \nabla u(t) \|_p$ with $p > m + 2$

We continue estimations for the solutions $u(t)$.

Proposition 2.4. *We have the estimate*

$$\| \nabla u(t) \|_p \leq C(M_0, \| u_0 \|_2) t^{-\mu}, \qquad p > m + 2 \tag{2.14}$$

with a certain $\mu > 0$.

Proof. Multiplying the equation (1.1) by $-div\{|\nabla u|^{p-2}\nabla u\}$, $p \geq m + 2$, and integrating by parts, we have

$$\frac{1}{p}\frac{d}{dt} \| \nabla u \|_p^p + \int_\Omega \{\sigma \nabla u\}_i \{|\nabla u|^{p-2}\nabla u\}_j dx \;\; = -\int_\Omega (g' u_j + f_j)|\nabla u|^{p-2} u_j dx.$$

Here, to treat the second term of the left-hand side we further integrate by parts. Then,

$$\begin{aligned}
&\int_\Omega \{\sigma\nabla u\}_i \{|\nabla u|^{p-2}\nabla u\}_j dx \\
&\quad = -\int_\Omega \{\sigma \nabla u\}_{ij}\{|\nabla u|^{p-2}\nabla u\} dx + \int_{\partial\Omega} \{\sigma\nabla u\}_i \{|\nabla u|^{p-2}\nabla u\} n_j dS \\
&\quad = \int_\Omega \{2\sigma' u_k u_i u_{k,j} u_{i,j} + \sigma u_{ij}\}|\nabla u|^{p-2} u_{ij} \; + \; \frac{p-2}{4}\int_\Omega \sigma |\nabla u|^{p-4}|\nabla(|\nabla u|^2)|^2 dx \\
&\quad + \int_{\partial\Omega} |\nabla u|^{p-2}\{\sigma u_{ii} u_j n_i + 2\sigma' u_k u_i u_{kj} u_j n_i - \sigma u_{ij} u_j n_j\} n_j - 2\sigma' u_k u_i u_{kj} u_j n_j dS \\
&\quad = \;\; \int_\Omega |\nabla u|^{p-2}(\sigma u_{ij}^2 + 2\sigma' u_k u_i u_{kj} u_{ij}) dx \\
&\quad + \;\; \frac{p-2}{4}\int_\Omega \{\sigma |\nabla u|^{p-4}|\nabla(|\nabla u|^2)|^2 \{2\sigma'(\textstyle\sum_{j,k} u_j u_k u_{jk})^2 + \sigma \sum_i (\sum_j u_i j u_j)^2\} dS \\
&\quad + \;\; \int_{\partial\Omega} |\nabla u|^{p-2}\sigma \frac{\partial u}{\partial n}(\Delta u - \frac{\partial^2 u}{\partial n^2}) dS, \\
&\quad \geq \;\; k_0 (\int_\Omega |\nabla u|^{p-2}|D^2 u|^2 dx + \frac{p+m-4}{4}\int_\Omega |\nabla u|^{p+m-4}\nabla(|\nabla u|^2)|^2 dx) \\
&\quad -C(N-1)\int_{\partial\Omega} |\nabla u|^{p+m} dS,
\end{aligned}$$

where we have used the fact that $H(x) = (\frac{\partial u}{\partial n})^{-1}(\Delta u - \frac{\partial^2 u}{\partial n^2})$ is the mean curvature on $\partial\Omega$. We see for $p \geq m + 2$,

$$\int_\Omega |g_u u_j + f_j||\nabla u|^{p-2}|u_j| dx \leq C(\| \nabla u \|_p^p + M_1 \| \nabla u \|_p^{p-1}). \tag{2.15}$$

Thus, we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt} \| \nabla u(t) \|_p^p + \frac{C_1}{p} \| \| |\nabla u| \|_{1,2}^{(p+m)/2} \\
&\qquad \leq \frac{C_1}{p}\int_\Omega |\nabla u|^{p+m} dx + C\int_{\partial\Omega} |\nabla u|^{p+m} + C_2(\| \nabla u \|_p^p + M_0 \| \nabla u \|_p^{p-1}).
\end{aligned} \tag{2.16}$$

By a standard trace theorem and Gagliardo-Nirenberg inequality, the boundary integral on the fourth term in (2.16) can be estimated as

$$\int_{\partial\Omega} |\nabla u|^{p+m} dS \leq C \| |\nabla u|^{(p+m)/2} \|_{H^{1/2}}^2 \leq \| \nabla u|^{(p+m)/2} \|_2 \| \nabla u|^{(p+m)/2} \|_{H^1}$$

$$\leq \frac{1}{4}\frac{C_1}{p} \parallel \left|\nabla u\right|^{\frac{p+m}{2}} \parallel_{1,2}^2 + Cp \int_\Omega \left|\nabla u\right|^{p+m} dx. \qquad (2.17)$$

To estimate the second term of the right hand side of (2.16) we make the following device

$$p \parallel \nabla u \parallel_{p+m}^{p+m} \leq \parallel \nabla u \parallel_{m+2}^{\theta_1(p+m)} \parallel \nabla u \parallel_p^{\theta_2(p+m)} \parallel |\nabla u|^{(p+m)/2} \parallel_{1,2}^{2\theta_3}$$

$$\leq \frac{1}{4}\frac{C_1}{p} \parallel |\nabla u|^{\frac{p+m}{2}} \parallel_{1,2}^2 + Cp^2 \parallel \nabla u \parallel_{m+2}^m \parallel \nabla u \parallel_p^p, \qquad (2.18)$$

where $\theta_i, i = 1,2,3$ are chosen in such a way that

$$\theta_1 + \theta_2 + \theta_3 = 1,$$

$$\frac{1}{p+m} = \frac{\theta_1}{m+2} + \frac{\theta_2}{p} + \frac{2\theta_3}{r}, r \leq \frac{2N}{(N-2)^+},$$

$$(p+m)\theta_2 = (1-\theta_3)p.$$

If $N \geq 3$, we take

$$\theta_1 = \frac{2m(m+2)}{(mN+2m+4)}, \theta_2 = \frac{2p(m+2)}{(p+m)(mN+2m+4)}, \theta_3 = \frac{(p+m)mN}{(p+m)(mN+2m+4)}.$$

From (2.16) - (2.18) we obtain

$$\frac{1}{p}\frac{d}{dt} \parallel \nabla u(t) \parallel_p^p + \frac{C_1}{p} \parallel |\nabla u|^{\frac{p+m}{2}} \parallel_{1,2}^2 \leq C(M_1)p^3 t^{-m\lambda} \parallel \nabla u(t) \parallel_p^p + CM_1 \parallel \nabla u|_p^{p-1},$$

with $0 < t \leq 1$. (2.19)

Now, by Lemma 2.1, we have

$$\parallel \nabla u \parallel_p \leq C^{1/p} \parallel \nabla u \parallel_{m+2}^{1-\theta} \parallel |\nabla u|^{(p+m)/2} \parallel_{1,2}^{2\theta/(m+2)}$$

with

$$\theta = \frac{p+m}{2}\left(\frac{1}{m+2} - \frac{1}{p}\right)\left(\frac{1}{N} - \frac{1}{2} + \frac{p+m}{2m+4}\right)^{-1}.$$

Hence,

$$\parallel \nabla u|^{(p+m)/2} \parallel_{1,2}^2) \parallel \geq C_1 - 2/\eta \parallel \nabla u \parallel_{m+2}^{(\eta-1)(p+m)/\eta} \parallel \nabla u \parallel_p^{p+m}.$$

Thus we have from (2.19),

$$\frac{d}{dt} \parallel \nabla u(t) \parallel_p^p + CC_1^{-2/\theta} \parallel \nabla u \parallel_{m+2}^{(p+m)(1-1/\theta)} \parallel \nabla u \parallel_p^{(p+m)/\theta}$$

$$\leq Cp^3 t^{-m\lambda} \parallel \nabla u(t) \parallel_p^p + CM_0 \parallel \nabla u|_p^{p-1}, \quad 0 < t \leq 1.$$

By Young inequality we get

$$\frac{d}{dt} \parallel \nabla u(t) \parallel_p^p + 2^{-1}CC_1^{-2/\theta} t^{-\mu(p+m)(1-1/\theta)} \parallel \nabla u \parallel_p^{(p+m)/\theta}$$

$$\leq Cp^3 t^{-m\lambda} \parallel \nabla u(t) \parallel_p^p + CM_{10}^p, \quad 0 < t \leq 1. \qquad (2.20)$$

Applying Lemma 2.2 to (2.20) we obtain

$$\parallel \nabla u(t) \parallel_p \leq C(p, M_0)t^{-\mu}, \quad 0 < t \leq 1$$

with a certain $\mu > 0$. ■

Proposition 2.5. *We have the estimate*

$$\| \nabla u(t) \|_p \leq C(p, M_0) < \infty, t \geq 1, \quad p > m + 2 \qquad (2.21)$$

Proof. Let $t \geq 1$. In this case we use

$$\nabla u \|_p \leq C^{1/p} \| \nabla \|_{m+2}^{1-\theta} \| |\nabla u|^{(p+n)/2} \|_{1,2}^{2\theta/(p+m)}$$

with a certain θ and instead of (2.18),

$$p \| \nabla u \|_{p+m}^{p+m} \leq \frac{1}{4} \frac{C_1}{p} \| |\nabla u|^{(p+m)/2} \|_{1,2}^2 + Cp^2 \| \nabla u \|_{m+2}^{p+m} \qquad (2.22)$$

to obtain, instead of (2.19)

$$\frac{1}{p} \frac{d}{dt} \| \nabla u(t) \|_p^p + \frac{C_1}{p} \| |\nabla u|^{\frac{p+m}{2}} \|_{1,2}^2 \leq C(M_0) p, \ t \geq 1 \qquad (2.23)$$

Applying Lemma 2.2 we have

$$\| \nabla u(t) \|_p \leq C(p, M_0) < \infty, t \geq 1. ■$$

References

[1] N. Alikakos, R. Rostamian, Gradient estimates for degenerate diffusion quations, *Math. Ann.*, 259, p. 53–70, 1982.

[2] Engler, H., Kawohl, B. and Luckhaus, S, Gradient estimates for solutions of parabolic equations and systems, *J. Math. Anal. Appl.*, Vol. 147, p.309–329. 1990.

[3] J. W. Cholewa, T. Dlotko, *Global Attractors in Abstract Parabolic Problems*, Cambridge Univ. Press., Cambridge, 2000.

[4] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, AMS, Providence,RI, 1988.

[5] A. Haraux, *Nonlinear Evolution Equations-Global Behavior of Solutions*, Lect. Notes in Math. 841, Springer, 1981.

[6] M. Nakao, L^p estimates of solutions of some nonlinear degenerate diffusion equations, *J. Math. Soc. Japan* **37**, p.41-63, 1985.

[7] M. Nakao, N. Aris, Global attractor for a quasilinear parabolic equations of mean curvature type, *International Journal of Dynamical Systems and Differential Equations*, Vol. 1, No. 1, p.75–84, 2008.

[8] M. Nakao, C. Chen, Global existence and gradient estimate for the quasilinear parobolic equations of m-Laplacian type with a nonlinear convection term, *J. Diff. Eqns*. **162**, p.224-250, 2000.

[9] M. Nakao, Y. Ohara, Gradient estimate of periodic solution for some quasilinear parobolic equations , *J.Math.Anal.Appl*. **204**, p.868-883, 1996.

[10] R.A. Adams, *Sobolev Space*, Academic Press, New York, 2000.